Remarks concerning the Expansions of the Hyperelliptic Sigma-Functions.

By Oskar Bolza.

The following remarks are supplementary to my two papers, "The Partial Differential Equations for the Hyperelliptic Θ- and $\mathfrak{S}$-Functions," and "Proof of Brioschi's Recursion Formula for the Expansion of the Even $\mathfrak{S}$-Functions of Two Variables," American Journal of Mathematics, vol. XXI, pp. 107–125, and pp. 175–190. I shall refer to them as B. I and B. II respectively, and shall use the same notation as there.[*]

If
$$R(x) = \phi(x)\,\psi(x) \tag{1}$$

be a decomposition of $R(x)$ into two factors of degree $\rho + 1$, then[†]
$$F_0(x, \xi) = \tfrac{1}{2}(\phi(x)\,\psi(\xi) + \phi(\xi)\,\psi(x)) \tag{2}$$

is one of the possible forms of Weierstrass' function $F(x, \xi)$, since it satisfies the conditions (2) of B. I.

Hence equation (41) of B. I becomes
$$\sum_{\alpha,\,\beta} \left(\frac{\partial^2 \log \mathfrak{S}_0}{\partial u_\alpha\, \partial u_\beta} \right)_0 g_\alpha(s)\, g_\beta(t) \equiv 0 \tag{3}$$

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*The following misprints in these papers have come to my notice:

p. 109, line 14, read $x - a$ instead of $a - a$.

p. 115, line 12, read (17) instead of (27).

p. 118, line 4 from bottom, read $\overline{U} = U$, $\overline{\Gamma} = \Gamma$.

p. 122, foot-note, drop upper index (x).

p. 124, line 2, sign of first term on the right, $+$ instead of $-$.

p. 124, equation (41), read $(t - s)^2$ instead of $(t - s)$.

p. 176, equation (5), read y^2 instead of y_2.

p. 176, equation (9), read $\lambda_{11}\, x\xi$ instead of $\lambda_{11}\, x$.

p. 176, foot-note, add: (H), (6), (7), (10).

p. 177, lines 5 and 6, the sign $=$ ought to be on the right of the term following it.

p. 183, equation (18), read N_x^3 instead of N^3.

†See Baker, Amer. Journ. of Math., vol. XX, p. 337.

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for all values of s and t, if we denote by $\mathfrak{G}_0$ the $\mathfrak{G}$-function of algebraic character-
istic $\phi\psi$ associated with $F_0(x, \xi)$. Therefore, *the terms of dimension two will
disappear from the expansion of $\mathfrak{G}_0$ into a power series.*

A special interest attaches, therefore, to the function $\mathfrak{G}_0$, and I propose to
consider in the following note:

    1). The partial differential equations for $\mathfrak{G}_0$.

    2). The recursion formulæ for the expansion of $\mathfrak{G}_0$ for the lowest cases $\rho = 1$
and $\rho = 2$.

§1. *The Partial Differential Equation for $\mathfrak{G}_0$.*

THEOREM I.—*The $\mathfrak{G}$-function of algebraic characteristic $\phi\psi$ associated with the
function*

$$F_0(x, \xi) = \tfrac{1}{2} \left[\phi(x)\psi(\xi) + \phi(\xi)\psi(x) \right]$$

and denoted by $\mathfrak{G}_0$, satisfies the partial differential equation

$$\frac{\partial \mathfrak{G}_0}{\partial a} = -\tfrac{1}{2} \mathfrak{G}_0 \sum_{a,\beta} \lambda_{a\beta} u_a u_\beta - \sum_{a,\beta} \varkappa_{a,\beta} u_a \frac{\partial \mathfrak{G}_0}{\partial u_\beta} + \frac{1}{R'(a)} \sum_{a,\beta} \frac{\partial^2 \mathfrak{G}_0}{\partial u_a \partial u_\beta} g_a(a) g_\beta(a), \quad (4)$$

the coefficients $\lambda_{a\beta}$, $\varkappa_{a\beta}$ being defined by the equations

$$R'(a) \Lambda(x, \xi) \equiv R'(a) \sum_{a,\beta} \lambda_{a\beta} g_a(x) g_\beta(\xi) = \frac{\Phi(x, \xi)\Phi(a, a) - \Phi(x, a)\Phi(\xi, a)}{8(x-a)(\xi-a)}, \quad (5)$$

$$R'(a) \mathrm{K}(x, \xi) \equiv R'(a) \sum_{a,\beta} \varkappa_{a\beta} g_a(x) h_\beta(\xi)$$
$$= \frac{(x-\xi)^{\rho-1}\Phi(a, a) - (a-\xi)^{\rho-1}\Phi(x, a)}{2(x-a)}, \quad (6)$$

$$(x-\xi)^{\rho-1} = \sum_\beta g_\beta(x) h_\beta(\xi), \quad (7)$$

where

$$\Phi(x, \xi) = \frac{\phi(x)\psi(\xi) - \phi(\xi)\psi(x)}{(x-\xi)}. \quad (8)$$

Proof: a). As a consequence of (3), equation (40) of B. I becomes

$$\frac{\partial \log \Theta(0, 0, \ldots, 0)}{\partial a} = -2 \sum_{a,\beta} \frac{g_a(a) g_\beta(a) a_{a\beta}}{R'(a)},$$

and, therefore, the second term on the right-hand side of equation (H) of B. I
disappears.

b). The substitution of the special value (2) for $F(x, \xi)$ in B. I (7) and (10) furnishes

$$\Lambda(x, \xi) = \frac{\phi(x)\,\psi(\xi) - \phi(\xi)\,\psi(x)}{8\,(x - \xi)(x - a)(\xi - a)} - \frac{\phi(x)\,\phi(\xi)\,\psi^2(a)}{8\,R'(a)(x - a)^2(\xi - a)},$$

$$K(x, \xi) = \tfrac{1}{2}\frac{(x - \xi)^{\rho - 1}}{x - a} - \tfrac{1}{2}\frac{(a - \xi)^{\rho - 1}\phi(x)\,\psi(a)}{R'(a)(x - a)^2},$$

and if we notice that

$$\frac{\phi(x)\,\psi(a)}{x - a} = \Phi(x, a),$$

$$R'(a) = \Phi(a, a),$$

we obtain the above values (5) and (6).

From (4) follows immediately the

Corollary I:

If $\dfrac{T_n}{(2n)!}$ *denotes the term of order* $2n$ *in the expansion of* $\mathfrak{S}_0$ *according to powers of* $u_1, u_2, \ldots, u_\rho$, *then*

$$\frac{\partial T_{n-1}}{\partial a} = -(n - 1)(2n - 3)\,T_{n-2}\cdot\sum_{a, \beta}\lambda_{a\beta}\,u_a\,u_\beta$$

$$-\sum_{a, \beta}\varkappa_{a\beta}\,u_a\,\frac{\partial T_{n-1}}{\partial u_\beta} + \frac{1}{2n\,(2n - 1)\,R'(a)}\sum_{a, \beta}\frac{\partial^2 T_n}{\partial u_a\,\partial u_\beta}\,g_a(a)\,g_\beta(a). \quad (9)$$

From the derivative $\dfrac{\partial T_{n-1}}{\partial a}$ we can pass to any Aronhold-process by the

Corollary II:

If

$$M(x) = \sum_{i=0}^{\rho+1}\binom{\rho + 1}{i}\,M_i\,x^{\rho+1-i}$$

denotes an integral function of x *at most of degree* $\rho + 1$, *and if*

$$\phi(x) = \sum_{i=0}^{\rho+1}\binom{\rho + 1}{i}\,\phi_i\,x^{\rho+1-i},$$

then

$$\sum_{i=0}^{\rho+1}M_i\,\frac{\partial T_{n-1}}{\partial\phi_i} = (n - 1)\,\frac{M_0}{\phi_0}\,T_{n-1} - \sum_{(a)}\frac{M(a)}{\phi'(a)}\,\frac{\partial T_{n-1}}{\partial a}, \quad (10)$$

the summation $\displaystyle\sum_{(a)}$ *extending over the* $\rho + 1$ *roots of* $\phi(x)$.

Proof: $\sum\limits_{i=0}^{\rho+1} M_i \dfrac{\partial T_{n-1}}{\partial \phi_i}$ can be obtained by replacing, in $M(z)$, $\binom{\rho+1}{i} z^{\rho+1-i}$

by $\dfrac{\partial T_{n-1}}{\partial \phi_i}$, for $i = 0, 1, 2, \ldots, \rho + 1$. The same substitution changes

$$- \frac{\phi(z)}{z-a} \text{ into } \frac{\partial T_{n-1}}{\partial a} = \sum_{i=1}^{\rho+1} \frac{\partial T_{n-1}}{\partial \phi_i} \frac{\partial \phi_i}{\partial a},$$

since $\qquad \dfrac{\partial \phi(z)}{\partial a} = -\dfrac{\phi(z)}{z-a}$, say $= \sum\limits_{i=1}^{\rho+1} \binom{\rho+1}{i} B_i z^{\rho+1-i}$,

and, therefore,

$$\frac{\partial \phi_i}{\partial a} = B_i.$$

Now $\qquad M(z) - \dfrac{M_0}{\phi_0}\phi(z) = \sum_{(a)} \dfrac{M(a)}{\phi'(a)} \dfrac{\phi(z)}{z-a},$

and, therefore, by the above substitution

$$\sum_{i=0}^{\rho+1} M_i \frac{\partial T_{n-1}}{\partial \phi_i} = \frac{M_0}{\phi_0} \sum_{i=0}^{\rho+1} \phi_i \frac{\partial T_{n-1}}{\partial \phi_i} - \sum_{(a)} \frac{M(a)}{\phi'(a)} \frac{\partial T_{n-1}}{\partial a}.$$

But T_{n-1} is a homogeneous function[*] of $\phi_0, \phi_1, \ldots, \phi_{\rho+1}$ of dimension $n-1$, provided the $g_a(x)$'s are independent of these quantities; for the quantities

$$\omega_{a\lambda}, \quad \eta_{a\lambda}, \quad a_{a\beta}, \quad b_{a\beta}, \quad \tau_{a\beta}$$

are homogeneous in the ϕ_i's and of dimensions

$$-\tfrac{1}{2}, \quad +\tfrac{1}{2}, \quad +1, \quad +\tfrac{1}{2}, \quad 0$$

respectively. Hence, it follows from the definition of $\mathfrak{S}_0$ (cf. (18) and (39) of B. I) that

$$\mathfrak{S}_0(u_a;\; m\phi_i) = \mathfrak{S}_0(m^{\frac{1}{2}} u_a;\; \phi_i)$$

for every m, therefore,

$$\sum_{i=0}^{\rho+1} \phi_i \frac{\partial \mathfrak{S}_0}{\partial \phi_i} = \tfrac{1}{2} \sum_a u_a \frac{\partial \mathfrak{S}_0}{\partial u_a},$$

and, consequently, $\qquad \sum\limits_{i=0}^{\rho+1} \phi_i \dfrac{\partial T_{n-1}}{\partial \phi_i} = (n-1)\, T_{n-1},$

which completes the proof of (10).

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[*] Cf. Klein, "Hyperelliptische Sigmafunctionen II," Math. Ann., 32, p. 369. As I start from a different definition of the $\mathfrak{S}$-functions, I wished to give an independent proof.

The connection between Klein's $\mathfrak{S}$-function of characteristic $\phi\psi$ (denoted by $\mathfrak{S}$) and the function $\mathfrak{S}_0$ is stated in the following

Corollary III:

If we put $$F_0(x, \xi) - a_x^{\rho+1} a_\xi^{\rho+1}$$

(of which we know a priori that it is divisible by $(x - \xi)^2$), equal to

$$2(x - \xi)^2 \sum_{\alpha, \beta} d_{\alpha\beta}\, g_\alpha(x)\, g_\beta(\xi),$$

then $\mathfrak{S}$ and $\mathfrak{S}_0$ are connected by the relation

$$\mathfrak{S}_0 = e^{-\frac{1}{2} \sum\limits_{\alpha,\beta} d_{\alpha\beta}\, u_\alpha\, u_\beta}\, \mathfrak{S}. \qquad (11)$$

This follows immediately from the fact that $\mathfrak{S}$ is associated with

$$F(x, \xi) = a_x^{\rho+1} a_\xi^{\rho+1},$$

combined with equations (16), (25), (26) of my paper, " On Weierstrass' Systems of Hyperelliptic Integrals of the First and Second Kind," Papers read at the International Mathematical Congress, 1893.

From (11) follows that $\frac{1}{2}\sum\limits_{\alpha,\beta} d_{\alpha\beta}\, u_\alpha\, u_\beta$ is the term of the second order in the expansion of $\mathfrak{S}$ in confirmation of known results.[*]

§2.—*The Case* $\rho = 1$.

If we choose $g_1(x) = 1$ and put $T_n = c_n u^{2n}$, the differential equation (9) becomes

$$R'(a)\, \frac{\partial c_{n-1}}{\partial a} = \tfrac{1}{8}(n - 1)(2n - 3)\, R_{\phi\psi}\, c_{n-2} + (n - 1)\, \phi_0\, \psi(a)\, c_{n-1} + c_n, \quad (12)$$

$R_{\phi\psi}$ denoting the resultant of ϕ and ψ. For the computation of λ_{11}, notice that $\Lambda(x, \xi)$ is of degree $\rho - 1 = 0$ in x and ξ; hence, it can be computed by giving x and ξ any particular values. Putting $x = \xi = a'$, the second root of ϕ, we obtain

$$\lambda_{11} = \tfrac{1}{8}\, \frac{\phi'(a')\, \psi(a')}{(a - a')^2} = -\frac{\phi_0^2\, \psi(a)\, \psi(a')}{R'(a)} = -\tfrac{1}{8}\, \frac{R_{\phi\psi}}{R'(a)}.$$

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[*] Schröder, "Ueber den Zusammenhang der hyperelliptischen $\mathfrak{S}$- und ϑ-Functionen, Göttinger, Diss., 1890 (74), (85).

From (12) follows:

$$\sum_{(a)} R'(a) \frac{\partial c_{n-1}}{\partial a} = \frac{(n-1)(2n-3)}{4} R_{\phi\psi} c_n + 2c_n + (n-1)\phi_0 \sum_{(a)} \psi(a).$$

But

$$\sum_{(a)} \phi_0 \psi(a) = \sum_{(a)} \left(\phi_0 \psi(a) - \psi_0 \phi(a)\right) = 4\left[\vartheta_1 - \frac{\phi_1}{\phi_0}\vartheta_0\right],$$

$\vartheta_0, \vartheta_1, \vartheta_2$ being the coefficients of $\vartheta = (\phi, \psi)_1$.

On the other hand,

$$\sum_{(a)} R'(a) \frac{\partial c_{n-1}}{\partial a} = \sum_{(a)} \frac{\phi'^2(a)\psi(a)}{\phi'(a)} \frac{\partial c_{n-1}}{\partial a} = -2\Delta_\phi \sum_{(a)} \frac{\psi(a)}{\phi'(a)} \frac{\partial c_{n-1}}{\partial a},$$

Δ_ϕ being the discriminant of ϕ; but, according to (10),

$$-\sum_{(a)} \frac{\psi(a)}{\phi'(a)} \frac{\partial c_{n-1}}{\partial a} = \sum_{i=0}^{2} \psi_i \frac{\partial c_{n-1}}{\partial \phi_i} - (n-1)\frac{\psi_0}{\phi_0} c_{n-1},$$

hence,

$$\Delta_\phi \sum_{i=0}^{2} \psi_i \frac{\partial c_{n-1}}{\partial \phi_i} - (n-1)\frac{\psi_0}{\phi_0}\Delta_\phi c_{n-1}$$

$$= \frac{(n-1)(2n-3)}{8} R_{\phi\psi} c_{n-2} + c_n + 2(n-1)\left(\vartheta_1 - \frac{\phi_1}{\phi_0}\vartheta_0\right)c_{n-1}.$$

A similar equation is obtained by interchanging ϕ and ψ. Add the two equations and notice that

$$\frac{\psi_0}{\phi_0}\Delta_\phi + \frac{\phi_0}{\psi_0}\Delta_\psi + 2\vartheta_0\left(\frac{\psi_1}{\psi_0} - \frac{\phi_1}{\phi_0}\right) = 2A_{\phi\phi},$$

where $A_{\phi\psi} = (\phi, \psi)_2$.

The result is the following

Theorem II:

If $\mathfrak{S}_\lambda(u)$ is that even $\mathfrak{S}$-function which corresponds to the decomposition of the biquadratic $R(x)$ into the two quadratic factors $\phi\psi$, and if we write

$$\mathfrak{S}_\lambda(u) = e^{-\frac{1}{4}e_\lambda u^2}\left[1 + c_2\frac{u^4}{4!} + c_3\frac{u^6}{6!} + \ldots\right], \tag{13}$$

then the coefficients $c_2, c_3, \ldots$ are determined by the recursion-formula

$$c_n = \delta c_{n-1} - (n-1)A_{\phi\psi}c_{n-1} - \frac{(n-1)(2n-3)}{8}R_{\phi\psi}c_{n-2}, \tag{14}$$

where δ denotes the following Aronhold-process:

$$\delta f = \tfrac{1}{2} \left[\Delta_\phi \sum_{i=0}^{2} \psi_i \frac{\partial f}{\partial \phi_i} + \Delta_\psi \sum_{i=0}^{2} \phi_i \frac{\partial f}{\partial \psi_i} \right] \tag{15}$$

and
$$\Delta_\phi = (\phi, \phi)_2, \quad \Delta_\psi = (\psi, \psi)_2. \quad A_{\phi\psi} = (\phi, \psi)_2$$

and $R_{\phi\psi}$ denotes the resultant of ϕ and ψ.

Since $R_{\phi\psi}$ is a combinant of ϕ and ψ, we have at once

$$\delta R_{\phi\psi} = 0; \tag{16}$$

and the ordinary rules for Aronhold-process furnish easily

$$\delta A_{\phi\psi} = \Delta_\phi \Delta_\psi = A_{\phi\psi}^2 - R_{\phi\psi} . \tag{17}$$

Hence, follows the

THEOREM III :

The coefficient c_n in the expansion of $\mathfrak{S}_\lambda(u)$ are integral functions of the two invariants $A_{\phi\psi}$ and $R_{\phi\psi}$, and if we put

$$c_n = \sum_{i=0} \gamma_{n-2i}^{(n)} \quad R_{\phi\psi}^i \quad A_{\phi\psi}^{n-2i}, \tag{18}$$

with the agreement that $\gamma_{n-2i}^{(n)} = 0$ whenever $n - 2i < 0$, the coefficients $\gamma_{n-2i}^{(n)}$ are determined by the recursion-formulæ
$$\gamma_n^{(n)} = 0,$$
$$\gamma_{n-2i}^{(n)} = -2i\,\gamma_{n-1-2i}^{(n-1)} - (n+1-2i)\,\gamma_{n+1-2i}^{(n-1)} - \frac{(n-1)(2n-3)}{8}\,\gamma_{n-2i}^{(n-2)}. \tag{19}$$

The proof of (19) is immediate: substitute in (14) for c_n, c_{n-1}, c_{n-2} their expressions in terms of $A_{\phi\psi}$ and $R_{\phi\psi}$, apply (16) and (17), and equate corresponding terms on both sides.

To compare these results with those obtained by Weierstrass by an entirely different method (Werke, vol. II, p. 253), we have to take ϕ and ψ in the normal form :
$$\phi = 4z_2 (z_1 - e_\lambda z_2),$$
$$\psi = z_1^2 + e_\lambda z_1 z_2 + e_\mu e_\nu z_2^2,$$
which furnishes
$$A_{\phi\psi} = -6e_\lambda, \quad R_{\phi\psi} = 4(12e_\lambda^2 - g_2). \tag{20}$$

Weierstrass arranges c_n according to powers of
$$12e_\lambda = -2A_{\phi\psi},$$
and
$$2\varepsilon_\lambda = 6e_\lambda^2 - \tfrac{1}{2}g_2 = \tfrac{1}{8}R_{\phi\psi}.$$

Hence, in order to obtain accordance with Weierstrass' notation, we must replace

$$\gamma^{(n)}_{n-2i} \text{ by } \frac{(-2)^{n-2i}}{8^i} c_{n-2i,\,i},$$

after which substitution, formula (19) will exactly coincide with Weierstrass' recursion-formula (J).

§2.—*The Case* $\rho = 2$.

a). *Computation of* $\Lambda(x,\xi)$ *and* $\mathbf{K}(x,\xi)$.

For $\rho = 2$, $\Lambda(x,\xi)$ is the first polar of $\Lambda(x,x)$ with respect to ξ. According to (5),

$$R'(a)\,\Lambda(x,x) = \frac{\Phi(x,x)\,\Phi(a,a) - \Phi^2(x,a)}{8\,(x-a)^2};$$

but*
$$\left.\begin{aligned}\Phi(x,y) &= 3\vartheta_x^2\vartheta_y^2 + \tfrac{1}{2}J(xy)^2, \\ \text{where} \qquad \vartheta &= (\phi,\psi)_1, \quad J = (\phi,\psi)_3\end{aligned}\right\} \qquad (21)$$

and an easy symbolic computation, in which we have to make use of the relation*

$$J^2 = 6\,(\vartheta,\vartheta)_4,$$

leads to the result:

$$\Phi(x,x)\,\Phi(y,y) - \Phi^2(x,y) = 3\left[6H_x^2 H_y^2 - J\vartheta_x^2\vartheta_y^2\right](xy)^2,$$

where
$$H = (\vartheta,\vartheta)_2. \qquad (21)$$

The function $6H - J\vartheta$ plays an important part in the theory of the cubic involution

$$\lambda\phi + \mu\psi;$$

it is that biquadratic to which all the cubics of the involution are apolar. For shortness, we denote it by G:

$$6H - J\vartheta = G. \qquad (21)$$

Our result is then

$$R'(a)\,\Lambda(x,x) = \tfrac{3}{8}G_x^2 G_a^2. \qquad (22)$$

Hence, if we choose, as in B. II,

$$g_1(x) = \frac{x}{2}, \quad g_2(x) = \tfrac{1}{2}, \qquad (23)$$

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* A table of the principal formulæ concerning cubic involutions is given in §1 of my paper, " Die cubische Involution, etc." Math. Annalen, Bd. 50.

we have

$$R'(a) \sum_{a,\beta} \lambda_{a\beta} u_a u_\beta = \tfrac{3}{2} G_u^2 G_a^2. \tag{24}$$

Further, according to (6),

$$R'(a) \, \mathrm{K}(x, \xi) = \frac{(x - \xi) \, \Phi(a, a) - (a - \overset{\circ}{\xi}) \, \Phi(x, a)}{2(x - a)}$$

the right-hand side is an integral function of a of degree 3, which we denote by $P(a)$; the coefficient P_0 of a^3 is easily found to be

$$P_0 = -\tfrac{3}{2} \vartheta_0 (x - \xi).$$

Hence we may write

$$R'(a) \, \mathrm{K}(x, \xi) = P(a) + \tfrac{3}{2} \frac{\vartheta_0}{\varphi_0} (x - \xi) \varphi(a),$$

and the right-hand side is now reduced to the second degree in a.

According to the definition (6) of the $\varkappa_{a\beta}$'s, $\mathrm{K}(x, \xi)$ may be used as an abbreviation for

$$\sum_{a, \beta} \varkappa_{a\beta} u_a \frac{\partial T_{n-1}}{\partial u_\beta},$$

inasmuch as it changes into this expression, when we replace $g_a(x)$ by u_a, $h_\beta(\xi)$ by $\dfrac{\partial T_{n-1}}{\partial u_\beta}$. This substitution changes $(x - \xi)^{\rho-1}$ into

$$\sum_\beta u_\beta \frac{\partial T_{n-1}}{\partial u_\beta} = (2n - 2) T_{n-1}.$$

With the same agreement applied to $P(a)$, we may write our differential equation (9) in the form

$$\frac{\partial T_{n-1}}{\partial a} = \frac{1}{R'(a)} \left\{ - \frac{3(n-1)(2n-3)}{2} G_u^2 G_a^2 T_{n-2} \right.$$
$$\left. + \frac{1}{8n(2n-1)} \sum_{a,\beta} \frac{\partial^2 T_n}{\partial u_a \partial u_\beta} a^{4-a-\beta} - P(a) - \frac{3(n-1) \vartheta_0 \varphi(a)}{\varphi_0} T_{n-1} \right\}, \tag{25}$$

An analogous expression ($\overline{25}$) for the derivative of T_{n-1} with respect to a root b of ψ is derived from (25) by interchanging a and b, φ and ψ, which changes $G_u^2 G_a^2$ into $G_u^2 G_b^2$, $P(a)$ into $-P(b)$, ϑ_0 into $-\vartheta_0$.

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b). *The recursion-formula for* T_n.

In order to derive from (25) a recursion-formula for T_n, we apply the same method as in B. II, §2: Multiply (25) by $\dfrac{\phi(t)\,\psi(a)}{t-a}$, t denoting an arbitrary parameter, and add with respect to the roots of ϕ; multiply $(\overline{25})$ by $\dfrac{\psi(t)\,\phi(b)}{t-b}$ and add with respect to the roots of ψ. Finally, add the two results thus obtained.

Since the right-hand side of (25) is an integral function of the second degree of a, we can apply the reasoning of §2 of B. II and obtain as the result of the process on the right:

$$-3\,(n-1)(2n-3)\,G_u^2\,G_t^2\,T_{n-2} + \frac{1}{4n\,(2n-1)}\sum_{\alpha,\,\beta}\frac{\partial^2\,T_n}{\partial u_\alpha\,\partial u_\beta}\,t^{4-\alpha-\beta}$$

$$+\,\frac{3\,(n-1)\,\mathfrak{S}_0}{\phi_0\,\psi_0}\left(\phi_0\,\psi(t)-\psi_0\,\phi(t)\right)T_{n-1}.$$

The transformation of the left-hand side is given in §3 of B. II; but it can be very much simplified by the application of Corollary II of §1.

Since
$$\frac{\phi(t)\,\psi(a)}{t-a} = \Phi(t,a),$$

we may write

$$\sum_{(a)}\frac{\phi(t)\,\psi(a)}{t-a}\,\frac{\partial T_{n-1}}{\partial a} = \sum_{(a)}\frac{\phi'(a)\,\Phi(t,a)-\tfrac{3}{2}\,\phi(a)\,\Phi'(t,a)}{\phi'(a)}\,\frac{\partial T_{n-1}}{\partial a},$$

where $\Phi'(t,z) = \dfrac{\partial \Phi(t,z)}{\partial z}$ and the zero terms

$$-\tfrac{3}{2}\,\phi(a)\,\Phi'(t,a)$$

has been added to reduce the degree of the numerator to three.

Use symbolic notation and put

$$\Phi(t,z) = \tfrac{1}{2}\,\Phi_z^2, \tag{26}$$

then

$$\phi'(z)\,\Phi(t,z)-\tfrac{3}{2}\,\phi(z)\,\Phi'(t,z) = \tfrac{3}{2}\left[\phi_z^2\,\phi_1\,\Phi_z^2 - \phi_z^3\,\Phi_z\,\Phi_1\right] = -\tfrac{3}{2}\,(\Phi\phi)\,\Phi_z\,\phi_z^2.$$

Hence, if we put, as in B. II,

$$3\,(\Phi\phi)\,\Phi_z\,\phi_z^2 = M_x^3 \tag{27}$$

and apply (10), we obtain

$$\sum_{(a)} \frac{\phi(t)\,\psi(a)}{t-a}\,\frac{\partial T_{n-1}}{\partial a} = \tfrac{1}{2}\sum_{i=0}^{3} M_i\,\frac{\partial T_{n-1}}{\partial \phi_i} - \frac{(n-1)}{2}\,\frac{M_0}{\phi_0}\,T_{n-1}.$$

Similarly, if we put

$$-3\,(\Phi\psi)\,\Phi_z\,\psi_z^2 = N_z^3, \tag{27}$$

we have

$$\sum_{(b)} \frac{\psi(t)\,\phi(b)}{t-b}\,\frac{\partial T_{n-1}}{\partial b} = \tfrac{1}{2}\sum_{i=0}^{3} N_i\,\frac{\partial T_{n-1}}{\partial \psi_i} - \frac{(n-1)}{2}\,\frac{N_0}{\psi_0}\,T_{n-1}.$$

Hence, if we define the operator D by

$$D(f) = \sum_{i=0}^{3} \left(M_i\,\frac{\partial f}{\partial \phi_i} + N_i\,\frac{\partial f}{\partial \psi_i} \right), \tag{28}$$

and notice that

$$M_0\,\psi_0 + N_0\,\phi_0 = -3\Phi_0\,\vartheta_0,$$

we obtain on the left-hand side as final result of the above-described process:

$$\tfrac{1}{2} D(T_{n-1}) + \tfrac{3}{2}(n-1)\,\frac{\Phi_0\,\vartheta_0}{\phi_0\,\psi_0}\,T_{n-1}.$$

The second term cancels against the term

$$\frac{3(n-1)\,\vartheta_0}{\phi_0\,\psi_0}\,(\phi_0\,\psi(t) - \psi_0\,\phi(t))\,T_{n-1}$$

on the right. For if in the equation

$$\phi(z)\,\psi(t) - \phi(t)\,\psi(z) = \tfrac{1}{2}\,(zt)\,\Phi_z^2,$$

understood homogeneously, we put

$$z_1 = 1, \quad z_2 = 0; \quad t_1 = t, \quad t_2 = 1,$$

we obtain

$$\phi_0\,\psi(t) - \psi_0\,\phi(t) = \tfrac{1}{2}\,\Phi_0.$$

Hence, we have

$$D(T_{n-1}) = -6\,(n-1)(2n-3)\,G_u^2\,G_t^2\,T_{n-2}$$
$$+ \frac{1}{2n(2n-1)}\sum_{\alpha,\,\beta}\frac{\partial^2 T_n}{\partial u_\alpha\,\partial u_\beta}\,t^{4-\alpha-\beta}. \tag{29}$$

Replace t by $\dfrac{u_1}{u_2}$, multiply by u_2^2 and denote

$$u_2^2\big(D(f)\big)_{t=\frac{u_1}{u_2}} = D_0(f). \tag{30}$$

Make use of (11) and of the known value

$$\tfrac{9}{10}\,\Theta\,(u) = \tfrac{9}{10}\,(\phi\,\psi)^2\,\phi_u\,\psi_u \tag{31}$$

for the second term in the expansion of Klein's $\mathfrak{S}$-function, then the following result[*] is obtained:

THEOREM IV.

The successive terms in the expansion of Klein's $\mathfrak{S}$-function $\mathfrak{S}_{\phi\psi}\,(u_1\,u_2)$ into the series

$$\mathfrak{S}_{\phi\psi}\,(u_1,\,u_2) = e^{\tfrac{9}{10}\,\Theta\,(u)}\left\{1 + \frac{T_2}{4!} + \frac{T_3}{6!} + \ldots\right\} \tag{31}$$

are obtained by the recursion-formula

$$T_n = D_0\,(T_{n-1}) + 6\,(n-1)(2n-3)\,G\,(u)\,T_{n-2}, \tag{32}$$

where the operator D_0, the quadratic $\Theta\,(u)$, and the biquadratic $G\,(u)$ are defined by equations (21), (26) *to* (28), (30) *and* (31).

UNIVERSITY OF CHICAGO, *December 2d,* 1899.

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[*] Other recursion-formulæ for $\mathfrak{S}_0\,(u_1,\,u_2)$ have been given by Brioschi, "Sulla teorica delle funzioni iperellittiche di primo ordine," Cap. VIII, Annali di Matematica (2), vol. 14, in particular p. 314; and Wiltheiss, "Differentialgleichungen und Reihenentwicklungen der Θ-functionen," §6, Math. Annalen, Bd. 29, in particular p. 298.